Colorado Teacher-Authored Instructional Unit Sample

Mathematics
High School – Algebra 2

Unit Title: Logarithmic Log Jams

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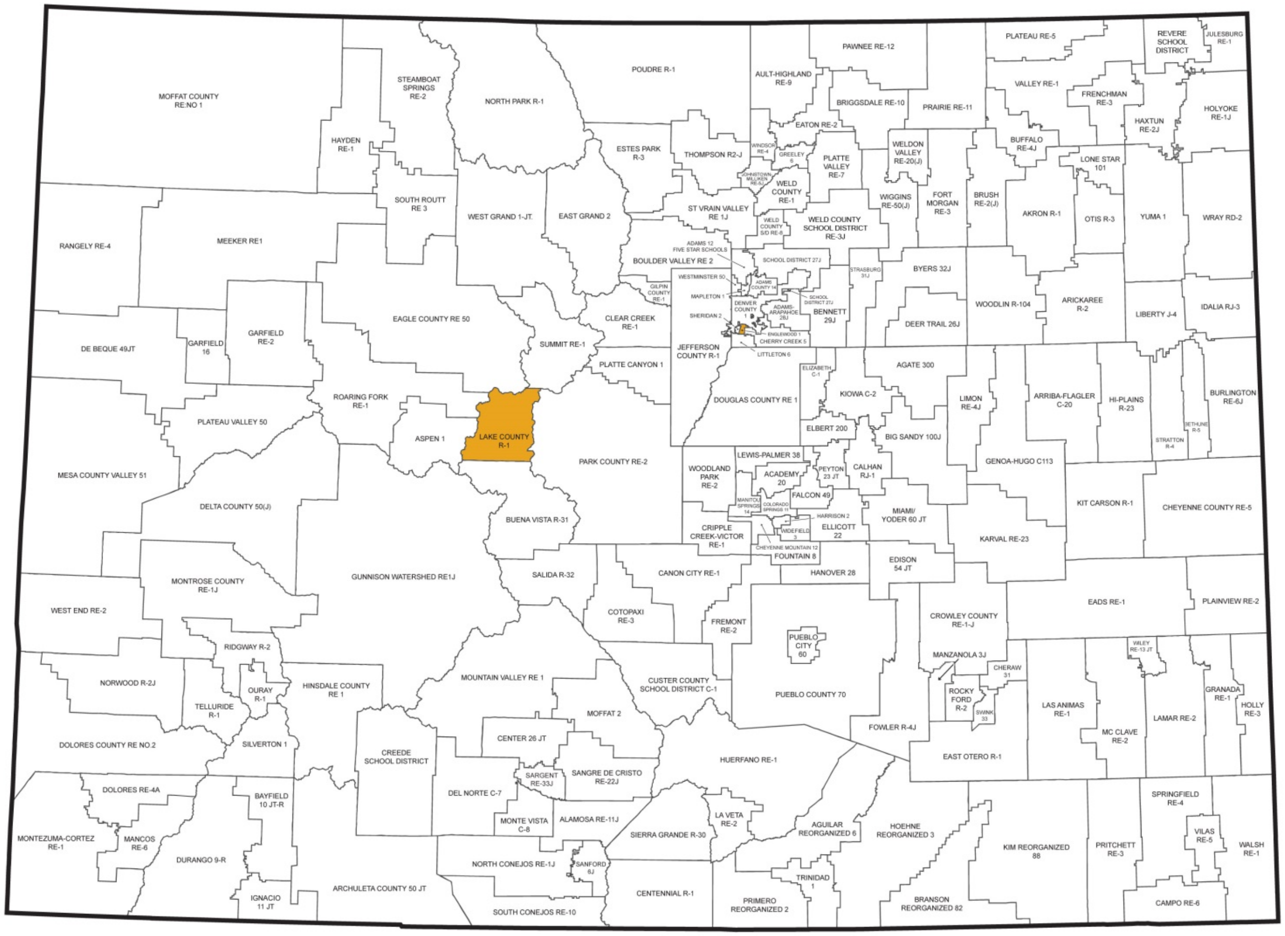

Colorado's District Sample Curriculum Project

This unit was authored by a team of Colorado educators. The template provided one example of unit design that enabled teacher-authors to organize possible learning experiences, resources, differentiation, and assessments. The unit is intended to support teachers, schools, and districts as they make their own local decisions around the best instructional plans and practices for all students.

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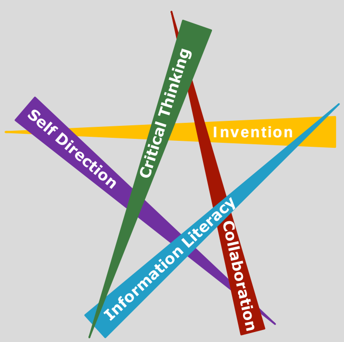
Colorado Teacher-Authored Sample Instructional Unit

Content Area	Mathematics	**Grade Level**	High School
Course Name/Course Code	Algebra 2		

Standard	**Grade Level Expectations (GLE)**	**GLE Code**
1. Number Sense, Properties, and Operations	1. The complex number system includes real numbers and imaginary numbers	MA10-GR.HS-S.1-GLE.1
	2. Quantitative reasoning is used to make sense of quantities and their relationships in problem situations	MA10-GR.HS-S.1-GLE.2
2. Patterns, Functions, and Algebraic Structures	1. Functions model situations where one quantity determines another and can be represented algebraically, graphically, and using tables	MA10-GR.HS-S.2-GLE.1
	2. Quantitative relationships in the real world can be modeled and solved using functions	MA10-GR.HS-S.2-GLE.2
	3. Expressions can be represented in multiple, equivalent forms	MA10-GR.HS-S.2-GLE.3
	4. Solutions to equations, inequalities and systems of equations are found using a variety of tools	MA10-GR.HS-S.2-GLE.4
3. Data Analysis, Statistics, and Probability	1. Visual displays and summary statistics condense the information in data sets into usable knowledge	MA10-GR.HS-S.3-GLE.1
	2. Statistical methods take variability into account supporting informed decisions making through quantitative studies designed to answer specific questions	MA10-GR.HS-S.3-GLE.2
	3. Probability models outcomes for situations in which there is inherent randomness	MA10-GR.HS-S.3-GLE.3
4. Shape, Dimension, and Geometric Relationships	1. Objects in the plane can be transformed, and those transformations can be described and analyzed mathematically	MA10-GR.HS-S.4-GLE.1
	2. Concepts of similarity are foundational to geometry and its applications	MA10-GR.HS-S.4-GLE.2
	3. Objects in the plane can be described and analyzed algebraically	MA10-GR.HS-S.4-GLE.3
	4. Attributes of two- and three-dimensional objects are measurable and can be quantified	MA10-GR.HS-S.4-GLE.4
	5. Objects in the real world can be modeled using geometric concepts	MA10-GR.HS-S.4-GLE.5

Colorado 21st Century Skills



Critical Thinking and Reasoning: *Thinking Deeply, Thinking Differently*

Information Literacy: *Untangling the Web*

Collaboration: *Working Together, Learning Together*

Self-Direction: *Own Your Learning*

Invention: *Creating Solutions*

Mathematical Practices:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Unit Titles	**Length of Unit/Contact Hours**	**Unit Number/Sequence**
Logarithmic Log Jams	4 Weeks	2

Unit Title	Logarithmic Log Jams		**Length of Unit**	4 Weeks
Focusing Lens(es)	Finance Growth	**Standards and Grade Level Expectations Addressed in this Unit**	MA10-GR.HS-S.2-GLE.1 MA10-GR.HS-S.2-GLE.2 MA10-GR.HS-S.2-GLE.3 MA10-GR.HS-S.2-GLE.4	
Inquiry Questions (Engaging-Debatable):	• What is the best way of paying of debt on multiple credit cards? • What financial phenomena can be modeled with exponential and linear functions? (MA10-GR.HS-S.2-GLE.2-IQ.3)			
Unit Strands	Algebra: Creating Equations Algebra: Seeing Structure in Expressions Functions: Interpreting Functions Functions: Building Functions Functions: Linear, Quadratic, and Exponential Models			
Concepts	Logarithms, inverse, exponential functions, growth, properties of exponents, properties of operations, expressions			

Generalizations **My students will Understand that…**	**Guiding Questions** **Factual**	**Conceptual**
Logarithms, the inverse of exponential functions, provide a mechanism for transforming and solving exponential functions. (MA10-GR.HS-S.2-GLE.1-EO.e) and (MA10-GR.HS-S.2-GLE.2-EO.a.iv)	What is the relationship of the graph of an exponential function and its inverse? How can you use the properties of exponents to represent an exponential function as a logarithm?	How are logarithms used to solve exponential functions? Why are logarithms inverses of exponential functions? (MA10-GR.HS-S.2-GLE.1-IQ.3)
Mathematicians derive exponential functions to model exponential growth. (MA10-GR.HS-S.2-GLE.3-EO.b)	What situation would be modeled by an exponential inequality? How are patterns and functions similar and different? (MA10-GR.HS-S.2-GLE.1-IQ.5)	Why is a geometric series modeled with an exponential function?
Properties of exponents and operations can transform expressions for exponential functions to facilitate interpretation of the quantities represented by the expression. (MA10-GR.HS-S.2-GLE.1-EO.c.)	What is the impact on the graph of transforming an expression?	Why might it be necessary to transform an exponential expression to better interpret the context of situation?

Key Knowledge and Skills: **My students will…**	*What students will know and be able to do are so closely linked in the concept-based discipline of mathematics. Therefore, in the mathematics samples what students should know and do are combined.*

- Create equations and inequalities in one variable and use them to solve problems. (MA10-GR.HS-S.2-GLE.4-EO.a.i)
- Use the properties of exponents to transform expressions for exponential functions with both rational and real exponents. (MA10-GR.HS-S.2-GLE.3-EO.b.i.3)
- Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. (MA10-GR.HS-S.2-GLE.3-EO.b.ii)
- For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology. (MA10-GR.HS-S.2-GLE.2-EO.a.iv)
- Graph exponential and logarithmic functions, showing intercepts and end behavior. (MA10-GR.HS-S.2-GLE.1-EO.c.iv)
- Use the properties of exponents to interpret expressions for exponential functions. (MA10-GR.HS-S.2-GLE.2-EO.c.v.2)
- Analyze the impact of interest rates on personal financial plans. PFL (MA10-GR.HS-S.2-GLE.2-EO.d.i)*
- Evaluate the costs and benefits of credit. PFL (MA10-GR.HS-S.2-GLE.2-EO.d.ii)*
- Analyze various lending sources, service and financial institutions. PFL (MA10-GR.HS-S.2-GLE.2-EO.d.iii)*

*** Denotes connection to Personal Financial Literacy (PFL)**

Critical Language: includes the Academic and Technical vocabulary, semantics, and discourse which are particular to and necessary for accessing a given discipline.
EXAMPLE: A student in Language Arts can demonstrate the ability to apply and comprehend critical language through the following statement: *"Mark Twain exposes the hypocrisy of slavery through the use of satire."*

A student in ______________ can demonstrate the ability to apply and comprehend critical language through the following statement(s):	*I know how to use properties of exponents to transform an exponential equation to a logarithm.*
Academic Vocabulary:	Graph, interpret, analyze, evaluate, solve, crate, formulas, equivalent, exponents, finite, growth, decay
Technical Vocabulary:	Logarithms, exponential functions, growth, properties of exponents, properties of operations, expressions, geometric series, inverse functions, intercepts, end behavior, geometric sequence, explicit, recursive, discrete, continuous, derive, common ratio

Unit Description:	This unit focuses on the exponential functions that are embedded in our everyday financial lives. Students explore credit card debt, interest rates, and the time it takes for investments to double in value. As they explore these topics, they are motivated to solve exponential equations, necessitating the introduction of logarithms. The students become fluent in rewriting expressions using logarithms and in solving for unknown variables in the exponent using logarithms. In addition, the students explore continuously compounded interest and discover one of the most important numbers in mathematics, e.
Unit Generalizations	
Key Generalization:	Logarithms, the inverse of exponential functions, provide a mechanism for transforming and solving exponential functions.
Supporting Generalizations:	Mathematicians derive exponential functions to model exponential growth.
	Properties of exponents and operations can transform expressions for exponential functions to facilitate interpretation of the quantities represented by the expression.

Performance Assessment: *The capstone/summative assessment for this unit.*	
Claims: (Key generalization(s) to be mastered and demonstrated through the capstone assessment.)	Logarithms, the inverse of exponential functions, provide a mechanism for transforming and solving exponential functions.
Stimulus Material: (Engaging scenario that includes role, audience, goal/outcome and explicitly connects the key generalization)	You would like to start a small business in your town selling handmade fleece clothing. In order to start the business, you will need to take out a loan of $200,000 (at 4% interest) to purchase equipment and supplies. However, the bank would like to see that you will be successful before giving you the loan. Specifically, the bank would like to see a projection of your expected profit. Your current estimate is that you will make $5000 in revenue each month with $3000 in expenses (not including your loan payment). You expect that your revenue could grow between 3% and 10% each year over the next decade while your expenses could grow between 1% and 5%. You need to model your profit for various scenarios of growth in revenue and expenses. In addition, you need to propose to the bank how long you would like the term of the loan to be (anywhere from 5 to 30 years) and factor how much this will cost into your profit projection. Remember, what you have left for profit is what you will live on. If you are only making $100 in profit each month, you must live on $100 a month. Your business will need a checking account through which all of your money will flow. You will also need a credit card in order to cover expenses that come up suddenly. Please include these in your business plan as well. You job is to put together a business plan with all of the above information so that the bank can approve your loan and you can get your business up and running.
Product/Evidence: (Expected product from students)	Students will produce a business plan that includes the following: 1. A choice of a business credit card and an explanation of why that was the best business credit card to choose, which should include information about both APR and fees. 2. A choice of a business checking account from a local bank and an explanation of why that was the best business checking

	account to choose, which should include information about fees and any interest earned. 3. A term for the loan and corresponding calculations for how much the loan will cost to pay back in total and how much it will cost you each month. 4. A worst-case projection that determines when your business might start to lose money, students will need to setup exponential functions and use logarithms to determine the details of this projection, unless using online calculator tools. 5. A best-case projection that determines when your business will start to make $20,000 per month. 6. A middle-ground projection that shows what the most likely growth for your business will be.
Differentiation: (Multiple modes for student expression)	Students can express their interpretations verbally through a video, presentation, or infographics. Students can work in collaborative groups on the project building on one another's strengths. Students can extend the analysis to examine what types of insurance might be necessary to purchase in order to transfer or minimize the risks associated with starting a small business.

Texts for independent reading or for class read aloud to support the content	
Informational/Non-Fiction	**Fiction**
The Young Investor by Katherine R Bateman *Straight Talk About Money* by Rendon, Marion B & Kranz, Rachel *Real World Math: Money & Other Numbers in Your Life* by Donna Guthrie	*Make Lemonade* by Virginia Wolff

Ongoing Discipline-Specific Learning Experiences				
1.	Description:	Think/work like a mathematician – Expressing mathematical reasoning by constructing viable arguments, critiquing the reasoning of others	Teacher Resources:	http://www.insidemathematics.org/index.php/standard-3 (examples of constructing viable arguments) http://quizlet.com/22134361/cpm-index-cards-of-teaching-strategies-flash-cards/ (teaching strategies to encourage class discussions)
			Student Resources:	N/A
	Skills:	Provide justification for arguments through a series of logical steps while using correct mathematical vocabulary. Analyze and critique the arguments of other students	Assessment:	Students justify their reasoning about exponential functions and classifications of numbers. Students use precise language such as exponential, rate of growth, common difference, rational, and irrational. Students can also critique the reasoning of others.
2.	Description:	Think/work like a mathematician – Engaging in the practice of modeling the solution to real world problems	Teacher Resources:	http://www.corestandards.org/Math/Content/HSM (Common Core State Standards description of the modeling process) http://blog.mrmeyer.com/?p=16301 (Dan Meyer discussion on modeling) http://threeacts.mrmeyer.com (Examples of 3-act problems)
			Student	N/A

			Resources:	
	Skills:	Model real world problems mapping relationships with appropriate models of functions, analyze relationships to draw conclusions, interpret results in relation to context, justify and defend the model, and reflect on whether results make sense	Assessment:	Modeling Problems Students use exponential and linear functions to model real world contexts with an emphasis on financial contexts. Students will be able to draw conclusions and interpret their models in relation to the context to determine if their model makes sense.
3.	Description:	Mathematicians are fluent with translating between recursive definitions and closed forms of sequences and series.	Teacher Resources:	http://www.shmoop.com/common-core-standards/ccss-hs-f-if-3.html (lesson plans for sequences)
			Student Resources:	http://www.shmoop.com/common-core-standards/ccss-hs-f-if-3.html (practices with sequences) http://www.classzone.com/etest/viewTestPractice.htm?ignoreRandom=true&testId=31757 (practice multiple choice questions for exponential and logarithmic functions)
	Skills:	Translating between recursive definitions and closed forms of sequences and series, including fitting functions to tables and problems in finance.	Assessment:	Fluency Problems Students build fluency with sequences and series particularly logarithms and exponential functions with consistent practice, including financial contexts.

Prior Knowledge and Experiences

Students studied inverse functions in the first unit of Algebra II (Functional Form and Design). The knowledge of how to find an inverse from a table and from a graph play an important role in helping students understand the relationship between exponential functions and logarithmic functions, which are inverses of one another. This unit also assumes that students are familiar with exponential functions, which was a main focus in Algebra I.

Learning Experience # 1: Geometric Series and Compound interest

The teacher may introduce a scenario to students where a person put $500 into an investment account that pays 5% interest each year for 50 years so that the students can determine how much they could save over the course of their lives by merely investing a small percentage of their income each year.

Enactive: Students can estimate how much money they think they could save over their life if they put $500 into an investment account each year and earned a 5% annual return.
Iconic: Students can create a table in excel that helps them to sum the total amount of money saved in the above scenario.
Symbolic: Students can derive the formula for summing a geometric series by summing how much each $500 placed in the account is worth at the end of the year and recognizing that it is a geometric series. Students can then use the formula for the sum of a geometric series to determine how much they could save over 50 years for varying amounts saved per year and varying interest rates.

Teacher Notes:	The teacher may want to give examples of what the rate of return on investments in the stock market typically earn over the course of 20 or 30 years. The teacher may also want to note that these rates of return are never guaranteed and often vary wildly from year to year, even if the average is somewhat constant over time.	
Generalization Connection(s):	Mathematicians derive exponential functions to model exponential growth.	
Teacher Resources:	https://www.illustrativemathematics.org/content-standards/tasks/1797 (geometric series task about the viewing of a u-tube video)	
Student Resources:	https://www.khanacademy.org/math/precalculus/seq_induction/geometric-sequence-series/e/geometric-series (contextualized practice questions about a geometric series) https://www.khanacademy.org/math/precalculus/seq_induction/geometric-sequence-series/e/geometric-series--1 (practice questions without a context for a geometric series)	
Assessment:	Students mastering the concept and skills of this lesson should be able to answer questions such as: • Why is a geometric series modeled with an exponential function? • Where does the formula for the sum of a geometric series come from?	
Differentiation: (Multiple means for students to access content and multiple modes for student to express understanding.)	**Access** (Resources and/or Process)	**Expression** (Products and/or Performance)
	Teacher may provide the steps for derive the formula for summing a geometric series.	Students can justify/explain each step in the process of deriving the formula for the summing of a geometric series.
Extensions for depth and complexity:	**Access** (Resources and/or Process)	**Expression** (Products and/or Performance)
	https://www.khanacademy.org/math/precalculus/seq_induction/geometric-sequence-series/v/geometric-series-sum-to-figure-out-mortgage-payments (Kahn video that shows mortgage rates from the perspective of a geometric series)	Students can explain how a mortgage calculator works based on geometric series.
Key Knowledge and Skills:	• Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems.	
Critical Language:	Rate of return, interest rate, investment, exponential, geometric series	

Learning Experience # 2: The Exponential Growth of Credit Card Debt
The teacher may have students brainstorm why people use credit cards so that the students can understand that there are times when using a credit card can be a benefit. The teacher may then provide students with examples of credit card interest rates and have the student calculate how much their credit card debt would grow over time if they did not pay it off so that students have an idea of the cost of using borrowed money. As a conclusion, the teacher may have the students write a letter to a fictional 16-year old son or daughter explaining how credit cards work and explaining why they think their fictional son or daughter should or should not get a credit card when they turn 18 so that the students can evaluate the

costs and benefits of credit. *Iconic:* Students can generate a table of values for different years given a starting amount of debt and an annual interest rate. Students can graph the growth of credit card debt over time. *Symbolic:* Students can express the formula for the growing credit card debt using variables such as P for principle, i for interest rate, and t for number of years as an exponential function.		
Teacher Notes:	The teacher should, at this point in the unit, use annual interest rates, even though credit cards typically use daily interest rates. Later, in learning experience 5, the students will be introduced to daily and monthly interest rates and can revisit the topic of APR as necessary. The teacher can decide whether to include information on credit history in the introduction to credit cards because one benefit of using a credit card at a young age is to build one's credit history. This exercise is both an introduction to the personal finance topic of credit cards and a review of exponential growth topics from Algebra I.	
Generalization Connection(s):	Mathematicians derive exponential functions to model exponential growth.	
Teacher Resources:	http://illuminations.nctm.org/Activity.aspx?id=3568 (applet that graphs various savings rates, credit card interest and stock market)	
Student Resources:	http://www.themint.org/teens/credit-card-facts.html (academic vocabulary for credit cards)	
Assessment:	Students mastering the concept and skills of this lesson should be able to answer questions such as: • Why does the debt on a credit card grow exponentially? • What are the reasons you may want to have a credit card? Why might you not want to have a credit card?	
Differentiation: (Multiple means for students to access content and multiple modes for student to express understanding.)	**Access** (Resources and/or Process)	**Expression** (Products and/or Performance)
	https://wvde.state.wv.us/strategybank/FrayerModel.html (Frayer model template for vocabulary words)	Student can write their letter using vocabulary support such as Frayer model cards.
Extensions for depth and complexity:	**Access** (Resources and/or Process)	**Expression** (Products and/or Performance)
	http://www.investopedia.com/university/teaching-financial-literacy-teens/teaching-financial-literacy-teens-credit-and-debt.asp (explanation of what is a FICO score and what impacts it)	Students can calculate the amount of additional money a car loan will cost if they have a 600, 700, or 800 FICO score.
Key Knowledge and Skills:	• Create equations and inequalities in one variable and use them to solve problems. • Evaluate the costs and benefits of credit.	
Critical Language:	Interest rate, debt, credit card, exponential function	

Learning Experience # 3: Using inverses to Introduce Logarithms		
The teacher may give the students situations of credit card debt where the students have to solve for time (e.g., If I have $500 in credit card debt that is compounded annually at 15% interest, how long until I owe the credit card company $2000?) so that students are motivated to find a new mathematical tool for solving an equation where the unknown variable is in the exponent. The teacher may then introduce the idea of a logarithm as a method for rewriting exponential equations so that students can solve the equations using the properties of logarithms or technology. *Iconic:* Students can graph the exponential function [500(1.15)^t] and its inverse in order to find the solution graphically. Students can also create a table and the table of the inverse function in order to estimate the value of the solution. This motivates the idea of a logarithm as the inverse to an exponent. *Symbolic:* Students can use the notation of how to express the inverse of an exponent as a logarithm and practice rewriting exponential expressions as logarithms. The students can then use both the properties of logarithms and technology to solve financial questions where they must solve for time.		
Teacher Notes:	This learning experience introduces the idea of a logarithm for the first time. This is the key skill for the unit, and the students will need to practice in order for them to become fluent. Although the remainder of the unit will use the idea of logarithms as well, the teacher may want to include many different ways for students to practice fluency with logarithms scattered throughout the unit. A few concepts that will need to be introduced in conjunction with this learning experience include the change of base formula for logarithms and how to evaluate logarithms on a calculator.	
Generalization Connection(s):	Logarithms, the inverse of exponential functions, provide a mechanism for transforming and solving exponential functions.	
Teacher Resources:	http://www.shmoop.com/common-core-standards/ccss-hs-f-le-4.html (explanation of logarithms) http://www.mathematicsvisionproject.org/uploads/1/1/6/3/11636986/sec3_mod2_logfun_te_041214.pdf (practice with logarithms, such as graphing values on number lines and coordinate graphing) http://www.fisme.science.uu.nl/en/fius/rmeconference/2009/handouts/geist/transformation.pdf (powerpoint slides with ideas for teaching logarithms) http://www.shmoop.com/logarithm-exponent/ (funny story about logarithms)	
Student Resources:	https://learnzillion.com/lessonsets/438-express-solutions-to-exponential-models-as-logarithms-evaluate-using-technology (video about logarithms)	
Assessment:	Students mastering the concept and skills of this lesson should be able to answer questions such as: • How are logarithms used to solve exponential functions? • Why are logarithms inverses of exponential functions? • What is the relationship of the graph of an exponential function and its inverse? • How can you use the properties of exponents to represent an exponential function as a logarithm?	
Differentiation: (Multiple means for students to access content and multiple modes for student to express understanding.)	**Access** (Resources and/or Process)	**Expression** (Products and/or Performance)
	http://geogebra.org (graphing tool can find inverse functions symbolically and graphically using the invert command)	Students can describe the connections between the exponential functions and their inverses (logarithms) supported by graphing technology.

Extensions for depth and complexity:	**Access** (Resources and/or Process)	**Expression** (Products and/or Performance)
	http://sciencenetlinks.com/lessons/frosty-the-snowman-meets-his-demise/ (activity for logarithms connection to science and carbon dating)	Students can determine how long it will take for a snowman to melt by using logarithms and connect this work to carbon dating.
Key Knowledge and Skills:	• For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology. • Graph exponential and logarithmic functions, showing intercepts and end behavior.	
Critical Language:	Logarithm, exponential form, logarithmic form, inverse	

Learning Experience # 4

The teacher may provide the students with a variety of annual interest rates for a savings account so that students can determine how long until an investment of $1000 doubles in value and discover the doubling formula for investments. Students can finish this experience by explaining how different interest rates in their lives such as credit cards, savings accounts, CDs, car loans, or a mortgage will affect their personal financial decisions. Students can create a comparison of typical interest rates for all of the above so that they have an understanding of how interest rates typically vary among different types of debt instruments or investments.

Iconic: Students can work in small groups to calculate the doubling times for different interest rates and then the class can compile the data to look for a pattern between the interest rate and the amount of time it takes an investment to double in value.

Symbolic: Students can express the pattern as a formula such as: Time to Double = 70/r, where r is the interest rate.

Teacher Notes:	The doubling formula for interest rates is known as the Rule of 69, or the Rule of 70, or the Rule of 72. These are all approximations, and any of them can be used. The emphasis of this exploration is on looking for and finding a pattern in the time it takes for an investment to double, not on the derivation of the doubling formula mathematically. By having the students practice finding how long it takes to double, they will practice solving equations using logarithms.
Generalization Connection(s):	Properties of exponents and operations can transform expressions for exponential functions to facilitate interpretation of the quantities represented by the expression.
Teacher Resources:	http://www.financeformulas.net/Doubling_Time.html (doubling time calculator displaying the underlying logarithmic formula) http://www.purplemath.com/modules/investmt.htm (additional practice with investment word problems)
Student Resources:	http://illuminations.nctm.org/Activity.aspx?id=3568 (applet showing the graph for savings rate, credit card interest and stock market)
Assessment:	Students mastering the concept and skills of this lesson should be able to answer questions such as: • How do you determine how long it takes an investment to double in value? • How do you solve an exponential equation?

Differentiation: (Multiple means for students to access content and multiple modes for student to express understanding.)	**Access** (Resources and/or Process)	**Expression** (Products and/or Performance)
	http://www.financeformulas.net/Doubling_Time.html (doubling time calculator displaying the underlying logarithmic formula)	Students can find the doubling time pattern by using a doubling time calculator to generate numerous examples.
Extensions for depth and complexity:	**Access** (Resources and/or Process)	**Expression** (Products and/or Performance)
	http://www.albartlett.org/presentations/arithmetic_population_energy.html (exponential growth and policy decision making)	Students can write a presentation to policy makers about the importance of understanding exponential growth when creating legislation.
Key Knowledge and Skills:	• Analyze the impact of interest rates on a personal financial plan. • For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where *a*, *c*, and *d* are numbers and the base *b* is 2, 10, or *e*; evaluate the logarithm using technology.	
Critical Language:	Doubling formula, rate of return	

Learning Experience # 5: The Difference between Yearly, Monthly, and Daily Compound Interest	
The teacher may provide students with examples of annual percentage rates (APR) that are compounded daily, monthly, or yearly from savings accounts, credit cards, or other financial situations so that students can explore how to generate an exponential equation for daily or monthly compound interest. Students can then use the given APR to calculate the annual percentage yield (APY), which can be slightly different depending on how often the interest rate is compounded. *Enactive:* Students can explore credit card, mortgage, or savings account offers to see how the advertised interest rate (APR) may differ from the annual percentage yield (APY). *Iconic:* Students can graph the two different exponential functions using both an APY and an advertised APR to see the difference in how much the function grows over time. *Symbolic:* Students can manipulate exponential expressions so that given a value for the APY of a credit card or a savings account, the students can determine the corresponding monthly or daily interest rate. For example, given an APY of 33%, students can determine what the daily periodic interest rate would be. 1.33^t = [1.33^(1/365)]^(365t)=1.0007682^(365t), so the amount of interest accumulated each day would be .07682%.	
Teacher Notes:	This learning experience introduces more complicated interest rates as noted in learning experience 2. There are subtle differences between Annual Percentage Rate (APR) and Annual Percentage Yield (APY). There is a resource listed in the teacher resources that explains the difference between these two terms.
Generalization Connection(s):	Properties of exponents and operations can transform expressions for exponential functions to facilitate interpretation of the quantities represented by the expression.
Teacher Resources:	http://www.investopedia.com/articles/basics/04/102904.asp (difference between APR and APY) http://www.shmoop.com/common-core-standards/ccss-hs-a-sse-3c.html (overview of transforming exponential functions) https://www.illustrativemathematics.org/content-standards/HSA/SSE/B/3/tasks/1305 (carbon dating task that requires transforming the exponential function) https://learnzillion.com/lessons/3498-compare-exponential-models-using-properties-of-exponents (comparing exponential models using properties of exponents)

Student Resources:	https://www.desmos.com/calculator (graphing applet)	
Assessment:	Students mastering the concept and skills of this lesson should be able to answer questions such as: • How do banks determine daily periodic rate for a credit card or a savings account? • How do you calculate the amount of interest that you will be charged on your credit card each month? • Why does compounding interest daily lead to more interest than compounding interest annually? • What is the difference between APR and APY?	
Differentiation: (Multiple means for students to access content and multiple modes for student to express understanding.)	**Access** (Resources and/or Process)	**Expression** (Products and/or Performance)
	The teachers may provide a yearly and monthly formula.	Students can justify why the two formulas are equivalent (e.g., graphing) and explain how one formula can be transformed into the other.
Extensions for depth and complexity:	**Access** (Resources and/or Process)	**Expression** (Products and/or Performance)
	http://www.investopedia.com/articles/basics/04/102904.asp (difference between APR and APY)	Students can collect advertisements and use them to explain why some financial products advertise APR and some use APY.
Key Knowledge and Skills:	• Use the properties of exponents to transform expressions for exponential functions with both rational and real exponents.	
Critical Language:	Daily interest rate, daily periodic rate, annual percentage rate (APR), annual percentage yield (APY)	

Learning Experience # 6: Deriving e by Continuously Compounding Interest	
The teacher may provide students with scenarios where interest on a savings account is compounded more and more often (yearly, monthly, daily, hourly, every minute, every second, etc.) so that students can explore the limit of compounding at a particular interest rate. *Iconic:* Students can explore graphically and in tables how compounding more often leads to more money, but that the increase in interest income soon becomes something that makes almost no difference. This is in essence a limit—the limit on how much interest income one can earn at a specific interest rate, no matter how often the account is compounded. *Symbolic:* Students can explore what happens when an account is compounded an infinite number of times (compounded continuously) and derive the formula for compounding continuously (y=Pe^(rt)) as shown below in the teacher notes.	
Teacher Notes:	Much of the derivation of the number *e* given above uses the idea of a limit, and the teacher may choose to use this terminology or not during this exploration. The teacher may show students how to manipulate the expression y=P(1+r/n)^(nt), where n is the number of times the account is compounded each year, P is the principle, r is the interest rate, and t is the number of years the account is compounded. By first rewriting the expression as y=P[1+1/(n/r)]^[(n/r)(rt)] they can explore the limit (horizontal asymptote) of y=(1+1/(n/r)^(n/r) as n goes to infinity. This will be *e*, which is why the limit of the expression y=P(1+r/n)^(nt) is Pe^(rt). This is the first time that students have been introduced to the number *e*. The teacher may want to point out that *e* is analogous to pi, an irrational number that is discovered, not invented. With the introduction of *e*, the teacher may introduce the natural logarithm as well. The teacher may decide to have students spend some time graphing logarithmic functions of base *e*.

Generalization Connection(s):	Properties of exponents and operations can transform expressions for exponential functions to facilitate interpretation of the quantities represented by the expression.	
Teacher Resources:	http://betterexplained.com/articles/demystifying-the-natural-logarithm-ln/ (explanation about the natural logarithm) https://www.khanacademy.org/math/algebra2/exponential_and_logarithmic_func/continuous_compounding/v/introduction-to-compound-interest-and-e (Kahn video on natural logarithm)	
Student Resources:	https://www.desmos.com/calculator (graphing applet)	
Assessment:	Students mastering the concept and skills of this lesson should be able to answer questions such as: • What is the number *e*? • What is the natural logarithm? How does compounding continuously relate to e?	
Differentiation: (Multiple means for students to access content and multiple modes for student to express understanding.)	**Access** (Resources and/or Process)	**Expression** (Products and/or Performance)
	https://learnzillion.com/lessons/220-understand-and-apply-the-definition-of-irrational-numbers (reminder about the concept of irrational numbers)	Students can explain why *e*, just like ∏, is a quantity/number and not a variable.
Extensions for depth and complexity:	**Access** (Resources and/or Process)	**Expression** (Products and/or Performance)
	https://www.youtube.com/watch?v=TINfzxSnnIE (video that explores 0.999999…..=1)	Students can explain the similarity between the concept of *e* and the concept that 0.999999…..=1.
Key Knowledge and Skills:	• Use the properties of exponents to transform expressions for exponential functions with both rational and real exponents.	
Critical Language:	*e*, irrational, limit, horizontal asymptote, compound continuously	

Learning Experience # 7: Undoing, Solving, more complicated	
The teacher may provide a number of scenarios (from personal finance or other areas of science and math) so that the students can solve equations using logarithms, including the natural logarithms, in context and express their answers in the context of the scenario.	
Teacher Notes:	Students should both be able to express their answers both as an expression involving logarithms (exact answer) and as an approximation by evaluating the logarithm using technology. There are many different areas of math and science where logarithms are used, including but not limited to: carbon dating, radioactive decay, Richter scale, pH, etc. After a unit focused mostly on PFL the point of this learning experience is to expose students to other contexts for logarithms and exponential functions. This learning experience does not introduce new mathematics but rather provides opportunities for students to solidify their understandings on logarithms and exponential in a variety of contexts.
Generalization Connection(s):	Logarithms, the inverse of exponential functions, provide a mechanism for transforming and solving exponential functions.
Teacher Resources:	https://www.illustrativemathematics.org/HSF-LE.A.4 (logarithm tasks) http://www.montereyinstitute.org/courses/DevelopmentalMath/COURSE_TEXT2_RESOURCE/U18_L4_T2_text_final.html (logarithm

	tasks)	
Student Resources:	https://www.desmos.com/calculator (graphing applet)	
Assessment:	Students mastering the concept and skills of this lesson should be able to answer questions such as: • What is the most efficient way to solve an exponential equation? • How do you check to see if an answer is reasonable?	
Differentiation: **(Multiple means for students to access content and multiple modes for student to express understanding.)**	**Access (Resources and/or Process)**	**Expression (Products and/or Performance)**
	N/A	N/A
Extensions for depth and complexity:	**Access (Resources and/or Process)**	**Expression (Products and/or Performance)**
	N/A	N/A
Key Knowledge and Skills:	• For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.	
Critical Language:	Logarithm, exponential, carbon dating, radioactive decay, pH, Richter scale	